OBSTRUCTIONS TO FLATNESS VIA THE EQUIVALENCE METHOD

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1. THE COFRAME BUNDLE

One of the fundamental problems in geometry is that of equivalence, the problem of determining when two objects in a geometric category are isomorphic. Here I remark on geometries which are 'G-structural,' which roughly means that they can be defined by a reduction of the structure group of the tangent bundle. The choice of G determines a kind of geometry, in the sense that G is the group of 'local symmetries' of your geometry, or the group which preserves the framings compatible with a geometry. Preliminary to this, let us consider the 'trivial geometry' of smooth manifolds.

Definition 1. Given a smooth manifold M^n, a *coframe* at $x \in M$ is a linear isomorphism

$$u\colon T_xM \to \mathbb{R}^n\,.$$

The set $\mathcal{F}_x$ of coframes based at x is a $\mathrm{GL}(\mathbb{R}^n)$ torsor, with action given by postcomposition (by the inverse, for reasons of convention). The *frame bundle*

$$\mathcal{F}(M) = \cup_{x\in M}\mathcal{F}_x \subset \mathrm{Hom}(TM, \underline{\mathbb{R}}^n).$$

is a $\mathrm{GL}(\mathbb{R}^n)$-principal bundle with projection map π sending $u \in \mathcal{F}_x$ to x.

For frames $u, v \in \mathcal{F}_x$ the map $A = uv^{-1} \in \mathrm{GL}(\mathbb{R}^n)$ is the unique element of $\mathrm{GL}(\mathbb{R}^n)$ so that $Au = v$.

I'll drop the M in $\mathcal{F}(M)$ from here on in the notation. I'll also take this opportunity to make the blanket assumption that M is contractible, since we're only discussing local matters in this note.

Fix a basis of $\mathbb{R}^n$. A little thought shows that locally, a coframing of M (n independent 1-forms $\eta^1, \dots, \eta^n$) is equivalent to a section of $\mathcal{F}$.

Definition 2. The *tautological coframe form* on $\mathcal{F}$ is the $\mathbb{R}^n$-valued 1-form $\omega \in \Omega^1(\mathcal{F}, \underline{\mathbb{R}}^n)$ given by the formula

$$\omega(\vec{v}) = u(\pi'(\vec{v})) = (\pi^*u)(\vec{v}) \qquad \text{for} \qquad \vec{v} \in T_u\mathcal{F}.$$

The reason that this is called the tautological form is that it has the following reproducing property.

Proposition 1. *Given a section η of $\mathcal{F}$,*

$$\eta^*\omega = \eta.$$

Proof. Fix $\vec{v} \in T_xM$ and let $\eta(x) = u$. Note that $\eta'(\vec{v}) \in T_u\mathcal{F}$. Then

$$(\eta^*\omega)(\vec{v}) = \omega(\eta'(\vec{v})) = u(\pi'\eta'(\vec{v})) = u(\vec{v}) = \eta_x(\vec{v}).$$

□

This will let us treat all (eventually adapted to a geometry...) coframings at once.

2. G-STRUCTURES

Definition 3. Given $G \subset \mathrm{GL}(\mathbb{R}^n)$ a matrix Lie group, a *G-structure* on M^n is a principal G subbundle $\mathcal{B}$ of $\mathcal{F}$.

This is supposed to to be such that the sections are the coframing adapted to a certain specified geometric structure on M of a certain geometric type (the latter determining G.).

Example 1. Suppose M^{2n} has an almost complex structure J, so that J is an endomorphism of TM for which $J^2 = -I$ at each point. Give $\mathbb{R}^{2n}$ the canonical structure of $\mathbb{C}^n$ and consider any coframe $u \in \mathcal{F}$ as a map $u\colon T_xM \to \mathbb{C}^n$. Then u is *compatible* with J if $u(JX) = i\, u(X)$ for all $X \in T_{\pi(u)}M$. The subset $\mathcal{B}$ of frames compatible with J is a $GL(\mathbb{C}^n)$-structure on M. Indeed, we may write any two coframes in $\mathcal{B}$ as u and Au for some $A \in GL(\mathbb{R}^{2n})$, and then

$$iAu(X) = Au(JX) = Aiu(X).$$

Since this holds for all X, and u is surjective, we have $A \in GL(\mathbb{C}^n)$.

Conversely, given a principle $GL(\mathbb{C}^n)$ sub-bundle $\mathcal{B}$ of $\mathcal{F}$, we can determine a unique complex structure on M. For each point $x \in M$ we simply choose any frame $u \in \mathcal{B}_x$ and define J by requiring $u(Jv) = iu(v)$ for all $v \in T_xM$. This choice of J will not depend on our choice of u because any other coframe in $\mathcal{F}_x$ differs from u by an element of $GL(\mathbb{C}^n)$.

Note that the sections of $\mathcal{B}$ are in bijection with bases for $(T^\vee M)^{1,0}$.

Other examples:

Geometry	Group
Riemannian	$O(n)$
conformal	$CO(n)$
almost complex	$\mathrm{GL}(\mathbb{C}^n)$
almost symplectic	$SP(n)$
coframings	$\{Id\}$

Definition 4. A G-structure $\mathcal{B}$ is *flat* at x if there exists coordinates $x^1, \ldots, x^n\colon M \to \mathbb{R}^n$ about x so that the coframing

$$(\,dx^1, \ldots,\, dx^n)\colon TM \to \mathbb{R}^n$$

has image in $\mathcal{B}$.

Example 2. Let $\mathcal{B}$ the set of orthonormal coframes of a Riemannian manifold (M, g). Then $\mathcal{B}$ is flat if and only if M is (locally) isometric to $\mathbb{E}^n$, flat Euclidean space. Indeed, the coordinates $x^1, \ldots, x^n$ give an isometry from M to $\mathbb{E}^n$ if and only if $(\,dx^1, \ldots,\, dx^n)$ is an orthonormal coframing.

Proposition 2. *A given G-structure is flat at x if and only if there is an integral manifold $\eta\colon \Sigma \to \mathcal{B}$ of the exterior differential system*

$$(\mathcal{B}, \{\,d\omega^i, i = 1, \ldots, n\})$$

for which $\eta^*(\omega^1 \wedge \ldots \wedge \omega^n)$ never vanishes and so that $x \in \pi(\Sigma)$.

Proof. Suppose such $\eta\colon \Sigma \to \mathcal{B}$ exists. The condition that $\eta^*(\omega^1 \wedge \ldots \wedge \omega^n) \neq 0$ means that $\pi\eta\colon \Sigma \to M$ is a local diffeomorphism. Shrinking M and Σ if necessary, we may replace Σ with M. Then η is seen to be a section of $\mathcal{B}$. Since η is an integral embedding, for $i = 1, \ldots, n$,

$$0 = \eta^*(\,d\omega^i) = \,d(\eta^*\omega^i) = \,d\eta^i.$$

Thus, by the Poincaré lemma, there are functions x^i so that

$$\eta^i = \,dx^i.$$

Finally, since

$$dx^1 \wedge \ldots \wedge dx^n = \eta^1 \wedge \ldots \wedge \eta^n = \eta^*(\omega^1 \wedge \ldots \wedge \omega^n) \neq 0,$$

these x^i give flat coordinates.

The converse proceeds by checking that a flat section gives an integral manifold. $\square$

So far, we have simply reformulated a common problem in geometry, without proving much. Once we learn how to find solutions to EDS we will be able to prove when integral manifolds exist. However, we can already see obstructions to the existence of solutions without much more work.

Theorem 1 (Cartan's first structure equation)**.** *Given a G-structure $\mathcal{B}$, there is a locally a pseudo-connection*[1]

$$\varphi = (\varphi^i_j) \in \Omega^1(\mathcal{B}, \mathfrak{g})$$

and corresponding torsion function

$$T = (T^i_{jk}) \in \mathcal{C}^\infty(\mathcal{B}, (\Lambda^2 \mathbb{R}^n)^\vee \otimes \mathbb{R}^n)$$

so that

$$d\omega^i = -\varphi^i_j \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k. \tag{1}$$

Proof. (Tersely) The pseudo-connection φ may be constructed pointwise, so it is left to show that such a T exists.

The forms ω^i are a basis for the semi-basic forms[2] of $\mathcal{B}$. So it suffices to show that $d\omega^i + \varphi^i_j \wedge \omega^j$ is semi-basic.

One can check that under the action of an element $g \in G$ on $\mathcal{B}$, one has

$$g^*\omega = g^{-1}\omega$$

(the right hand side is the same postcomposition action as on $\mathcal{F}$.) Taking the derivative, for any $\tilde{X} = (\tilde{X}^i_j) \in \mathfrak{g}$,

$$-\tilde{X}^i_j \omega^j = \mathcal{L}_{\tilde{X}}\omega^i = \cancel{d(\tilde{X} \lrcorner \omega^i)} + \tilde{X} \lrcorner d\omega^i.$$

Finally, using the reproducing property of a pseudo-connection,

$$\tilde{X} \lrcorner (d\omega^i + \varphi^i_j \wedge \omega^j) = 0,$$

so that $d\omega^i + \varphi^i_j \wedge \omega^j$ is semi-basic. $\square$

Remark 1. The pseudo-connection φ^i_j is notably not unique! In fact, for any choice of function $(U^i_{jk}) \in \mathcal{C}^\infty(\mathcal{B}, \mathfrak{g} \otimes \mathbb{R}^n)$, the new form

$$(\tilde{\varphi}^i_j) = (\varphi^i_j + U^i_{jk}\omega^j)$$

will also be a pseudo-connection. In the process, T will also be modified:

$$\begin{aligned} d\omega^i &= (\varphi^i_j + U^i_{jk}\omega^k) \wedge \omega^j - U^i_{jk}\omega^k \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k \\ &= \tilde{\varphi}^i_j \wedge \omega^j + (T^i_{jk} + U^i_{[jk]})\omega^j \wedge \omega^k. \end{aligned}$$

Here $[jk]$ denotes anti-symmetrization in those indices.

Definition 5. If there is a choice of pseudo-connection so that $T(u) = 0$, then we say that the (first order) torsion of $\mathcal{B}$ is absorbable at u.

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[1] A pseudo-connection is almost a principal connection, but is not required to be equivariant in each fiber. It is $\mathfrak{g}$-valued and it satisfies the reproducing property $\varphi(\tilde{X}) = X$, where $\tilde{X}$ is the left invariant vector field corresponding to $X \in \mathfrak{g}$ under the principal bundle action on $\mathcal{B}$. Here $\mathfrak{g}$ is the Lie algebra of G.

[2] A form $\xi \in \Omega^*(\mathcal{B})$ is semi-basic if $v \lrcorner \xi = 0$ for any vertical vector $v \in T\mathcal{B}$. A vector is vertical if $\pi'(v) = 0$. Note that the $\tilde{X}$ of the previous footnote span the vertical vectors.

Fixing arbitrary φ and its corresponding torsion T^i_{jk}, the torsion of $\mathcal{B}$ is absorbable at u if and only if there is a choice of U^i_{jk} so that $T^i_{jk}(u) - U^i_{[jk]}(u) = 0$.

Note also that this absorbability depends on both $\mathfrak{g}$ (the geometry) and on T (a measure of how twisted the fibers of $\mathcal{B}$ are).

Now, define $\mathcal{B}_{abs}$ to be the locus of points in $\mathcal{B}$ where torsion is absorbable.

Proposition 3. *Any integral submanifold $\eta\colon M \to \mathcal{B}$ as in Proposition 2 must have image in B_{abs}.*

Proof. Since $\eta^1 \wedge \ldots \wedge \eta^n = \eta^*(\omega^1 \wedge \ldots \wedge \omega^n) \neq 0$, there are functions $U^i_{jk} \in \mathcal{C}^\infty(M)$ so that

$$\eta^* \varphi^i_j = U^i_{jk} \eta^i.$$

But then

$$\begin{aligned} 0 = \eta^*\, d\omega^i &= \eta^*(-\varphi^i_j \wedge \omega^j + T^i_{jk}\omega^j \wedge \omega^k) \\ &= -U^i_{jk}\eta^k \wedge \eta^j + \eta^* T^i_{jk}\eta^j \wedge \eta^k \\ &= (U^i_{[jk]} + \eta^* T^i_{jk})\eta^j \wedge \eta^k. \end{aligned}$$

Since the η^i are independent, this means that $(U^i_{[jk]} + \eta^* T^i_{jk}) = 0$. Hence the torsion is absorbed in the image of η for the pseudo-connection

$$(\varphi^i_j + \pi^* U^i_{jk}\omega^k).$$

□

Corollary 1. *If B_{abs} is too small (e.g of smaller dimension than M), then the EDS $(\mathcal{B}, \{\, d\omega^i \})$ has no solutions for which $\eta^*(\omega^1 \wedge \ldots \wedge \omega^n) \neq 0$, so $\mathcal{B}$ is not flat.*

In particular, if the torsion is nowhere absorbable, then $\mathcal{B}$ cannot be flat.